

Automorphic Lie Algebras and Cohomology of Root Systems

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Abstract

This paper defines a cohomology theory of root systems which emerges naturally in the context of Automorphic Lie Algebras (ALiAs) but applies more generally to deformations of Lie algebras obtained by assigning a monomial in a finite number of variables to each weight vector. In the theory of Automorphic Lie Algebras certain problems can be formulated and partially solved in terms of cohomology, in particular one can find explicit models for an ALiA in terms of monomial deformations of the original Lie algebra. In this paper we formulate a cohomology theory of root systems and define the cup product in this context; we show that it can be restricted to symmetric forms, that it is equivariant with respect to the automorphism group of the original Lie algebra, and finally we show acyclicity at dimension two of the symmetric part, which is exactly what is needed to find models for ALiAs explicitly.

AMS Subject Classification Numbers

17B05 Lie algebras and Lie superalgebras: Structure theory;
17B22 Lie algebras and Lie superalgebras: Root systems;
17B65 Lie algebras and Lie superalgebras: Infinite-dimensional Lie (super)algebras.

1 Introduction: deformations of Lie algebras

The purpose of this paper is to set up a cohomology theory for root systems Φ . The motivation comes from the theory of Automorphic Lie Algebras (ALiAs) [13, 12, 9], where one can compute a Cartan-Weyl basis [14, 10] and finds that the structure constants are monomials in the two variables \mathbb{I} and \mathbb{J} , the zero-homogeneous invariant functions (under the action of a polyhedral group on the Riemann sphere), with the relation $0 = 1 + \mathbb{I} + \mathbb{J}$. An immediate question is whether one can find a model for such a Lie algebra by taking the Weyl-Chevalley normal form of a classical Lie algebra over \mathbb{C} and multiply the weight vectors by monomials in \mathbb{I} and \mathbb{J} , as in Definition 1.3. Such a model can help us to solve the isomorphism question: consider different models of ALiAs and show that they are equivalent under the action of $\text{Aut}(\Phi)$, automorphisms of the root system. This model can be seen as a 1-form on the root system, in such a way that the coboundary operator d^1 of this 1-form determines the structure constants of the ALiA. A natural question is whether every ALiA

admits such a model: if the second cohomology is zero, this is indeed the case. The proof that the second cohomology is zero for root systems of simple Lie algebras is the main result of this paper. Since it is entirely constructive, the proof also provides an integration procedure, allowing one to find a model from the given ALiA.

The construction of a Lie algebra with parameters can also be done without reference to ALiAs, by changing the field to monomials with complex coefficients, that is to say, to each weight vector e_α one assigns a monomial. This description is worked out in this section and may serve as an example for the reader when we move on to the abstract theory in the subsequent sections.

Definition 1.1. Let $C_\wedge^2(\Phi, \mathbb{Z})$ be the space of skew 2-forms with arguments in the root system Φ and values in \mathbb{Z} , where ‘skew’ stands here for the property $\omega(\beta, \alpha) = -\omega(\alpha, \beta)$. It is well known (e.g. [7, Section 25.2], [15, Introduction]) that the bracket relations of a semisimple Lie algebra \mathfrak{g} over \mathbb{C} can be written in terms of a Cartan-Weyl basis $\langle e_\alpha, e_{-\alpha}, h_r \rangle_{\alpha \in \Phi^+, r=1, \dots, \ell}$, where Φ^+ is a set of positive roots, in which the commutation relations are:

$$\begin{aligned} [h_r, h_s] &= 0 \\ [h_r, e_\alpha] &= \alpha(h_r)e_\alpha \\ [e_\alpha, e_\beta] &= \varepsilon^2(\alpha, \beta)e_{\alpha+\beta}, \quad \varepsilon^2 \in C_\wedge^2(\Phi, \mathbb{Z}), \quad \alpha + \beta \in \Phi, \\ [e_\alpha, e_{-\alpha}] &= \varepsilon^2(\alpha, -\alpha)h_\alpha. \end{aligned}$$

We use the $+$, $-$ symbol for addition, resp. subtraction of the roots $\alpha \in \Phi$. If $\alpha = \sum_{i=1}^\ell m_i \alpha_i$, $\alpha_i \in \Delta$, where Δ denotes the set of simple roots, then $h_\alpha = \sum_{i=1}^\ell m_i h_i$. We scale $e_{-\alpha}$ such that $\varepsilon^2(\alpha, -\alpha) = 1$, $\alpha \in \Delta$.

For possible choices of the values of ε^2 see [15], and see [4, Section 03] and [8] for the early cohomological interpretation of it.

Remark 1.1. Notice that in $C_\wedge^2(\Phi, \mathbb{Z})$ the skew symmetry condition is $\omega(\beta, \alpha) = -\omega(\alpha, \beta)$, while in $C_-^2(\Phi, M)$ – see Section 4 – the antisymmetry condition will be defined by $\omega(\beta, \alpha) = \omega(\alpha, \beta)^{-1}$.

We now change the field \mathbb{C} in the Lie algebra by allowing the coefficients $\varepsilon^2(\alpha, \beta)$ to be multiplied by monomials in a finite number of variables and call the new Lie algebra $\tilde{\mathfrak{g}}$. An Automorphic Lie Algebra is of this form if its Cartan subalgebra equals that of the base Lie algebra with the field extended to the ring of automorphic functions (cf. [9]).

Definition 1.2. Let $M(\mathbb{I}_1, \dots, \mathbb{I}_k) = \{\mathbb{I}_1^{m_1} \dots \mathbb{I}_k^{m_k} \mid m_1, \dots, m_k \in \mathbb{Z}\}$ be the rational monomials, and let $M[\mathbb{I}_1, \dots, \mathbb{I}_k] = \{\mathbb{I}_1^{m_1} \dots \mathbb{I}_k^{m_k} \mid m_1, \dots, m_k \in \mathbb{N}_0\}$ be the natural monomials. The set of rational monomials (resp. natural monomials) inherits an additive and multiplicative structure from \mathbb{Z} (resp. \mathbb{N}) by identifying a monomial with its power. For instance: one adds $\mathbb{I}_1^4 \mathbb{I}_2^2$ to \mathbb{I}_1^3 to obtain $\mathbb{I}_1^7 \mathbb{I}_2^2$ and multiplies them to obtain \mathbb{I}_1^{12} . So we will identify $M(\mathbb{I}_1, \dots, \mathbb{I}_k)$ with \mathbb{Z}^k and $M[\mathbb{I}_1, \dots, \mathbb{I}_k]$ with \mathbb{N}_0^k .

We define a symmetric 2-form $\omega_+^2 \in C_+^2(\Phi, M(\mathbb{I}_1, \dots, \mathbb{I}_k))$, where the $+$ indicates the symmetry $\omega_+^2(\alpha, \beta) = \omega_+^2(\beta, \alpha)$. Together with the previous antisymmetric 2-form $\varepsilon^2 \in C_\wedge^2(\Phi, \mathbb{Z})$, we want it to define a Lie algebra by

$$\begin{aligned} [h_r, h_s] &= 0 \\ [h_r, e_\alpha] &= \alpha(h_r)e_\alpha \\ [e_\alpha, e_\beta] &= \varepsilon^2(\alpha, \beta)\omega_+^2(\alpha, \beta)e_{\alpha+\beta}, \quad \alpha + \beta \in \Phi, \\ [e_\alpha, e_{-\alpha}] &= \varepsilon^2(\alpha, -\alpha)\omega_+^2(\alpha, -\alpha)h_\alpha. \end{aligned} \tag{1}$$

For this to happen, the terms of the Jacobi identity must all three have the same monomial in

$M(\mathbb{I}_1, \dots, \mathbb{I}_k)$, i.e. computing modulo multiplication with integers one has:

$$\begin{aligned} [[e_\alpha, e_\beta], e_\gamma] &\equiv \omega_+^2(\alpha, \beta) \omega_+^2(\alpha + \beta, \gamma) e_{\alpha+\beta+\gamma} \\ [e_\alpha, [e_\beta, e_\gamma]] &\equiv \omega_+^2(\alpha, \beta + \gamma) \omega_+^2(\beta, \gamma) e_{\alpha+\beta+\gamma} \\ [e_\beta, [e_\alpha, e_\gamma]] &\equiv \omega_+^2(\beta, \alpha + \gamma) \omega_+^2(\alpha, \gamma) e_{\alpha+\beta+\gamma} \end{aligned}$$

where the multiplication among the ω_+^2 s is done according to the *addition* rules as given in Definition 1.2 for $M(\mathbb{I}_1, \dots, \mathbb{I}_k)$. We now define

$$d^2 \omega_+^2(\alpha, \beta, \gamma) = \frac{\omega_+^2(\beta, \gamma) \omega_+^2(\alpha, \beta + \gamma)}{\omega_+^2(\alpha, \beta) \omega_+^2(\alpha + \beta, \gamma)},$$

where d^2 is a coboundary operator – see Section 2.1 for this choice of coboundary notation.

If $d^2 \omega_+^2(\alpha, \beta, \gamma) = 1$ for all $\alpha, \beta, \gamma \in \Phi$ such that $\alpha + \beta$, $\beta + \gamma$ and $\alpha + \beta + \gamma$ exist, then the three monomials in the Jacobi identity are equal, and the Jacobi identity is satisfied (by the Jacobi identity of the underlying Lie algebra associated to Φ). Vice versa, if the Jacobi identity is satisfied, then $d^2 \omega_+^2(\alpha, \beta, \gamma) = 1$. Indeed, for every triple $\alpha, \beta, \gamma \in \Phi$ such that $\alpha + \beta$, $\beta + \gamma$ and $\alpha + \beta + \gamma$ are roots, the two terms $[[e_\alpha, e_\beta], e_\gamma]$ and $[e_\alpha, [e_\beta, e_\gamma]]$ in the Jacobi identity are nonzero. Therefore their monomial powers are identical, i.e. $d^2 \omega_+^2(\alpha, \beta, \gamma) = 1$.

Let $Z_+^2(\Phi, M(\mathbb{I}_1, \dots, \mathbb{I}_k))$ be the group of 2-cocycles. We have just proven the following theorem.

Theorem 1.1. *Any $\omega_+^2 \in Z_+^2(\Phi, M(\mathbb{I}_1, \dots, \mathbb{I}_k))$ determines a Lie algebra with monomial coefficients of the form (1).*

Definition 1.3. *Let $\omega^1 \in C^1(\Phi, M(\mathbb{I}_1, \dots, \mathbb{I}_k))$ and $\omega_+^2(\alpha, \beta) = d^1 \omega^1(\alpha, \beta) = \frac{\omega^1(\alpha) \omega^1(\beta)}{\omega^1(\alpha + \beta)}$. We say that ω^1 is a model for ω_+^2 . If we take a Cartan-Weyl basis of a Lie algebra and define $\tilde{e}_\alpha = \omega^1(\alpha) e_\alpha$, we again have a Lie algebra with an identical Cartan matrix.*

The previous definition is justified by the following corollary.

Corollary 1.1. *Let ω^1 be defined as above. Then $d^1 \omega^1 \in Z_+^2(\Phi, M(\mathbb{I}_1, \dots, \mathbb{I}_k))$ and satisfies the requirements to define a Lie algebra. Furthermore, $K_{\tilde{\mathfrak{g}}}(\tilde{e}_\alpha, \tilde{e}_{-\alpha}) = d^1 \omega^1(\alpha, -\alpha) K_{\mathfrak{g}}(e_\alpha, e_{-\alpha})$, $\alpha \in \Phi^+$, where $K_{\tilde{\mathfrak{g}}}$ and $K_{\mathfrak{g}}$ are the Killing forms of the Lie algebras $\tilde{\mathfrak{g}}$ and \mathfrak{g} , respectively.*

One of the fundamental questions is thus whether there is always a model. This is equivalent to the question whether the second cohomology group $H_+^2(\Phi, M(\mathbb{I}_1, \dots, \mathbb{I}_k))$ is trivial – see Section 6.

2 Cohomology

Let Φ be a reduced root system, that is a root system satisfying the additional property that the only multiples of a root α in Φ are $\pm\alpha$, and let $\Phi_0 = \Phi \cup \{0\}$. Let Φ^+ be a subset of positive roots and let Δ be the set of the corresponding simple roots, that is, positive roots which cannot be written as the sum of two elements of Φ^+ . We denote the addition in the root system by $+$ to discern it from the formal addition used in the definition of chains that is to follow. We define n -chains $T_n(\Phi)$ inductively as the set of expressions

$$[r_1 | r_2 | \dots | r_n], \quad r_j \in \Phi_0, \quad j = 1, \dots, n$$

such that

$$[r_1 | \dots | r_{i-1} | r_i + r_{i+1} | r_{i+2} | \dots | r_n] \in T_{n-1}(\Phi), \quad i = 1, \dots, n-1$$

with $T_1(\Phi) = \Phi_0$. This is done to avoid the possibility that in applying the boundary operator as defined in Section 2.1, we try to add roots the sum of which is not a root (and this again is related to the fact that for such roots the bracket of their weight vectors will be zero). In the formula above the chain is zero (denoted by $0_n \in T_n(\Phi)$) if one of its constituents is zero, as happens when opposite roots are added. The oldfashioned $|$ -notation (cf [3], later replaced by \otimes) is used here because the modern notation would seem to imply things like linearity with respect to $+$ which is not at all the case. The $[$ and $]$ are added for readability, usually when $n > 1$ but also sometimes in expressions like $[r_0 + r_1]$. Although we will use the same formulas as in group cohomology theory in the definition of the (co)boundary operators, it should be noticed that the root system Φ_0 is not a group, but a groupoid.

Let $C_n(\Phi)$ be the \mathbb{Z} -linear span of $T_n(\Phi)$ and $C^n(\Phi, M) = C^1(C_n(\Phi), M)$, where $C^1(\square, M)$ are the \mathbb{Z} -linear 1-forms with values in the \mathbb{Z} -module $M = R^k$, where R is a ring. Notice that in our applications to ALiAs (e.g. see Section 6.2) we would rather like to restrict to a semiring, \mathbb{N}_0 , but for the cohomological computations we will need a ring to compute in.

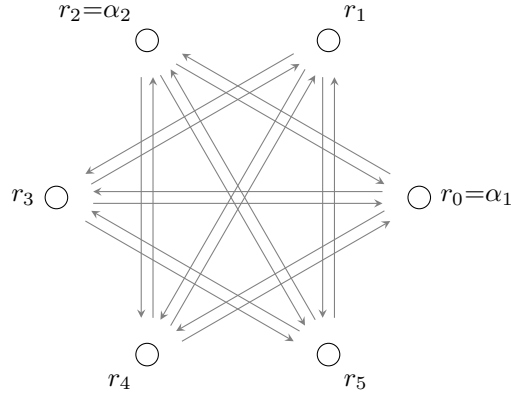
Example 2.1. *Let Φ be the root system of type A_2 . By abuse of notation we write $\Phi = A_2$. Let $\Delta = \{\alpha_1, \alpha_2\}$ and let $r_0 = \alpha_1$, $r_1 = \alpha_1 + \alpha_2$, $r_2 = \alpha_2$, $r_3 = -\alpha_1$, $r_4 = -\alpha_1 - \alpha_2$, $r_5 = -\alpha_2$. Then*

$$T_1(A_2) = \{0_1, r_0, r_1, r_2, r_3, r_4, r_5\}$$

and

$$T_2(A_2) = \{0_2, [r_0|r_2], [r_0|r_3], [r_0|r_4], [r_1|r_3], [r_1|r_4], [r_1|r_5], [r_2|r_0], [r_2|r_4], [r_2|r_5], [r_3|r_0], [r_3|r_1], [r_3|r_5], [r_4|r_0], [r_4|r_1], [r_4|r_2], [r_5|r_1], [r_5|r_2], [r_5|r_3]\}.$$

Figure 1: The basis for $C_2(A_2)$ depicted by gray arrows.



2.1 Differential

From here on everything is written additively, to increase readability, in contrast to the multiplicative notation used in Section 1. One can then define $\partial^n : T_{n+1}(\Phi) \rightarrow C_n(\Phi)$ in the usual manner

[6]. Here we give the first instances, followed by the general formula:

$$\begin{aligned}
\partial^0 r_0 &= 0 \\
\partial^1[r_0|r_1] &= r_1 - [r_0 + r_1] + r_0 \\
\partial^2[r_0|r_1|r_2] &= [r_1|r_2] - [r_0 + r_1|r_2] + [r_0|r_1 + r_2] - [r_0|r_1] \\
\partial^n[r_0|\dots|r_n] &= [r_1|\dots|r_n] + \sum_{j=1}^n (-1)^j [r_0|\dots|r_{j-1} + r_j|\dots|r_n] - (-1)^n [r_0|\dots|r_{n-1}].
\end{aligned}$$

It follows from the usual computation that $\partial^n \partial^{n+1} = 0$, that is ∂^n is a boundary operator. It can be extended by linearity to a map $\partial^n : C_{n+1}(\Phi) \rightarrow C_n(\Phi)$. Then the coboundary $d^n : C^n(\Phi, \mathbb{Z}^q) \rightarrow C^{n+1}(\Phi, \mathbb{Z}^q)$ is defined by $d^n \omega^n(r_0|\dots|r_n) = \omega^n(\partial^n[r_0|\dots|r_n])$ or, explicitly on the generators,

$$\begin{aligned}
d^0 \omega^0(r_0) &= 0 \\
d^1 \omega^1(r_0|r_1) &= \omega^1(r_1) - \omega^1(r_0 + r_1) + \omega^1(r_0) \\
d^2 \omega^2(r_0|r_1|r_2) &= \omega^2(r_1|r_2) - \omega^2(r_0 + r_1|r_2) + \omega^2(r_0|r_1 + r_2) - \omega^2(r_0|r_1) \\
d^n \omega^n(r_0|\dots|r_n) &= \omega^n(r_1|\dots|r_n) + \sum_{j=1}^n (-1)^j \omega^n(r_0|\dots|r_{j-1} + r_j|\dots|r_n) - (-1)^n \omega^n(r_0|\dots|r_{n-1}).
\end{aligned}$$

That $d^{n+1} d^n = 0$ follows from the definition and the fact that ∂^n is a boundary operator. We also want $\partial^n 0_{n+1} = 0_n$. Let us check that this is indeed the case: if $r_0 = 0$, then

$$\partial^n[0|r_1|\dots|r_n] = [r_1|\dots|r_n] - [0 + r_1|\dots|r_n] = 0_n.$$

If $r_k = 0, k = 1, \dots, n-1$ then

$$\partial^n[r_0|\dots|r_{k-1}|0|r_{k+1}|\dots|r_n] = (-1)^{k+1}[r_0|\dots|r_{k-1}+0|\dots|r_n] + (-1)^k[r_0|\dots|0+r_{k+1}|\dots|r_n] = 0_n$$

and if $r_n = 0$, then

$$\partial^n[r_0|\dots|r_{n-1}|0] = (-1)^n[r_0|\dots|r_{n-1}+0] - (-1)^n[r_0|\dots|r_{n-1}] = 0_n.$$

In the following sections we consider a number of concepts, namely cup product (Section 3), reversal symmetry (Section 4) and equivariance with respect to automorphisms of the root system (Section 5), before going to our main result, the acyclicity of the second cohomology in Section 6. These sections can all be read independently, with the exception of Section 4.1, as it depends on Section 3 and of Section 6, as it depends on Section 4.

3 Cup product

In this section the cup product is defined, giving $C^\bullet(\Phi, M)$ a ring structure.

Definition 3.1. *The cup product of two forms is defined by*

$$\omega^p \cup \omega^q(r_1|\dots|r_{p+q}) = \omega^p(r_1|\dots|r_p) \omega^q(r_{p+1}|\dots|r_{p+q}), \quad [r_1|\dots|r_{p+q}] \in T_{p+q}(\Phi).$$

Remark 3.1. *The multiplication here is the multiplication as defined in Definition 1.2 (or, more abstractly, if $M = R^k$, the multiplication in R). One has to multiply the powers of \mathbb{I} to get the right result: with $\alpha^1(r_0) = 1$ and $\alpha^1(r_1) = \mathbb{I}$, then $\alpha^1 \cup \alpha^1(r_0|r_1) = 1$, if $[r_0|r_1] \in T_2(\Phi)$, whereas $\alpha^1(r_0) + \alpha^1(r_1) = \mathbb{I}$ (the powers are added here).*

It is clear that $[r_1|\dots|r_p] \in T_p(\Phi)$ and $[r_{p+1}|\dots|r_{p+q}] \in T_q(\Phi)$ if $[r_1|\dots|r_{p+q}] \in T_{p+q}(\Phi)$, since there are less sums to check for existence in each of the partial terms.

Lemma 3.1. *The following product rule holds:*

$$\mathbf{d}^{p+q}\omega^p \cup \omega^q = \mathbf{d}^p\omega^p \cup \omega^q + (-1)^p\omega^p \cup \mathbf{d}^q\omega^q.$$

Proof.

$$\begin{aligned} & \mathbf{d}^{p+q}\omega^p \cup \omega^q(r_0 | \cdots | r_p | r_{p+1} | \cdots | r_{p+q}) \\ = & \omega^p(r_1 | \cdots | r_p)\omega^q(r_{p+1} | \cdots | r_{p+q}) - (-1)^{p+q}\omega^p(r_0 | \cdots | r_{p-1})\omega^q(r_p | \cdots | r_{p+q-1}) \\ & + \sum_{j=1}^p (-1)^j \omega^p(r_0 | \cdots | r_{j-1} + r_j | \cdots | r_p)\omega^q(r_{p+1} | \cdots | r_{p+q}) \\ & + \sum_{j=p+1}^{p+q} (-1)^j \omega^p(r_0 | \cdots | r_{p-1})\omega^q(r_p | \cdots | r_{j-1} + r_j | \cdots | r_{p+q}) \\ = & (-1)^p\omega^p(r_0 | \cdots | r_{p-1}) \left[-(-1)^q\omega^q(r_p | \cdots | r_{p+q-1}) \right. \\ & + \left. \sum_{j=1}^q (-1)^j \omega^q(r_p | \cdots | r_{j+p-1} + r_{j+p} | \cdots | r_{p+q}) + \omega^q(r_{p+1} | \cdots | r_{p+q}) \right] \\ & + \left[\omega^p(r_1 | \cdots | r_p) + \sum_{j=1}^p (-1)^j \omega^p(r_0 | \cdots | r_{j-1} + r_j | \cdots | r_p) \right. \\ & - \left. (-1)^p\omega^p(r_0 | \cdots | r_{p-1}) \right] \omega^q(r_{p+1} | \cdots | r_{p+q}) \\ = & \mathbf{d}^p\omega^p(r_0 | \cdots | r_p)\omega^q(r_{p+1} | \cdots | r_{p+q}) + (-1)^p\omega^p(r_0 | \cdots | r_{p-1})\mathbf{d}^q\omega^q(r_p | \cdots | r_{p+q}). \\ = & \mathbf{d}^p\omega^p \cup \omega^q(r_0 | \cdots | r_{p+q}) + (-1)^p\omega^p \cup \mathbf{d}^q\omega^q(r_0 | \cdots | r_{p+q}). \end{aligned}$$

□

Corollary 3.1. *The closed forms form a subring of $C^\bullet(\Phi, M)$ and the exact forms form an ideal within that subring.*

4 Reversal symmetry

Notice that when $[r_1 | r_2 | \cdots | r_n]$ is in $T_n(\Phi)$ then so is its opposite $\rho^n[r_1 | r_2 | \cdots | r_n] = [r_n | r_{n-1} | \cdots | r_1]$, so ρ^n is an involution on $T_n(\Phi)$. Let $\rho^n\omega^n(\phi) = \omega^n(\rho^n(\phi))$.

Definition 4.1. *Let $\kappa_n = \binom{n+1}{2} + 1$. Let $\hat{\rho}^n = (-1)^{\kappa_n}\rho^n$. We say that ω^n is symmetric if $\hat{\rho}^n\omega^n = \omega^n$ and antisymmetric if $\hat{\rho}^n\omega^n = -\omega^n$ (cf. Remark 1.1). Let $C_\pm^n(\Phi, M)$ consist of those $\omega^n \in C^n(\Phi, M)$ such that $\hat{\rho}^n\omega^n = \pm\omega^n$.*

Lemma 4.1 ([5, Section 3]). *One has $\hat{\rho}^{n+1}\mathbf{d}^n = \mathbf{d}^n\hat{\rho}^n$.*

Example 4.1.

$$\begin{aligned} \mathbf{d}^1\omega^1(r_0 | r_1) &= \omega^1(r_1) - \omega^1(r_0 + r_1) + \omega^1(r_0) \\ \hat{\rho}^2\mathbf{d}^1\omega^1(r_0 | r_1) &= \omega^1(r_1) - \omega^1(r_0 + r_1) + \omega^1(r_0) = \hat{\rho}^1(\omega^1(r_1) - \omega^1(r_0 + r_1) + \omega^1(r_0)) = \mathbf{d}^1\hat{\rho}^1\omega^1(r_0 | r_1) \\ \mathbf{d}^2\omega^2(r_0 | r_1 | r_2) &= \omega^2(r_1 | r_2) - \omega^2(r_0 + r_1 | r_2) + \omega^2(r_0 | r_1 + r_2) - \omega^2(r_0 | r_1) \\ \hat{\rho}^3\mathbf{d}^2\omega^2(r_0 | r_1 | r_2) &= -(\omega^2(r_1 | r_0) - \omega^2(r_2 + r_1 | r_0) + \omega^2(r_2 | r_1 + r_0) - \omega^2(r_2 | r_1)) \\ &= \hat{\rho}^2(-\omega^2(r_0 | r_1) + \omega^2(r_0 | r_2 + r_1) - \omega^2(r_1 + r_0 | r_2) + \omega^2(r_1 | r_2)) \\ &= \mathbf{d}^2\hat{\rho}^2\omega^2(r_0 | r_1 | r_2) \end{aligned}$$

Proof. Notice that $\kappa_n + \kappa_{n+1} = (n+1)^2 + 2 \equiv n+1 \pmod{2}$.

$$\begin{aligned}
\hat{\rho}^{n+1} \mathbf{d}^n \omega^n(r_0 | \cdots | r_n) &= (-1)^{\kappa_{n+1}} \left[\omega^n(r_{n-1} | \cdots | r_0) - (-1)^n \omega^n(r_n | \cdots | r_1) \right. \\
&\quad \left. + \sum_{j=1}^n (-1)^j \omega^n(r_n | \cdots | r_{n-j+2} | r_{n-j+1} + r_{n-j} | r_{n-j-1} | \cdots | r_0) \right] \\
&= (-1)^{\kappa_{n+1} + \kappa_n} \hat{\rho}^n \left[\omega^n(r_0 | \cdots | r_{n-1}) - (-1)^n \omega^n(r_1 | \cdots | r_n) \right. \\
&\quad \left. + \sum_{j=1}^n (-1)^j \omega^n(r_0 | \cdots | r_{n-j} + r_{n-j+1} | \cdots | r_n) \right] \\
&= (-1)^{n+1} \hat{\rho}^n \left[\omega^n(r_0 | \cdots | r_{n-1}) - (-1)^n \omega^n(r_1 | \cdots | r_n) \right. \\
&\quad \left. + \sum_{j=1}^n (-1)^{n-j+1} \omega^n(r_0 | \cdots | r_j + r_{j-1} | \cdots | r_n) \right] \\
&= (-1)^{n+1} (-1)^{n+1} \hat{\rho}^n \left[-(-1)^n \omega^n(r_0 | \cdots | r_{n-1}) + \omega^n(r_1 | \cdots | r_n) \right. \\
&\quad \left. + \sum_{j=1}^n (-1)^j \omega^n(r_0 | \cdots | r_{j-1} + r_j | \cdots | r_n) \right] \\
&= \mathbf{d}^n \hat{\rho}^n \omega^n(r_0 | \cdots | r_n).
\end{aligned}$$

This proves the statement. \square

Corollary 4.1. \mathbf{d}^n maps $C_{\pm}^n(\Phi, M)$ to $C_{\pm}^{n+1}(\Phi, M)$.

Notice in particular that $C^1(\Phi, M) = C_+^1(\Phi, M)$ and $H_-^2(\Phi, M) = Z_-^2(\Phi, M)$.

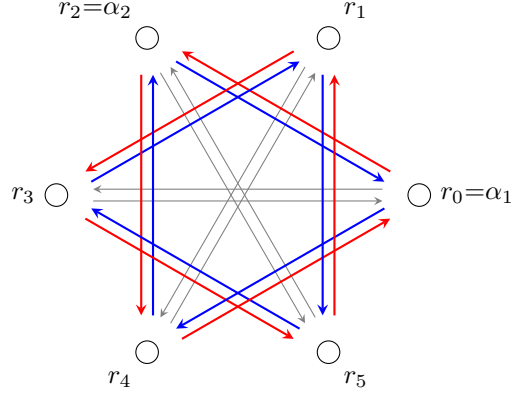
Example 4.2 ($H_-^2(A_2, M) \neq 0$). Let Φ be the root system of type A_2 . As in Example 2.1 we number the roots $r_0 = \alpha_1, r_1 = \alpha_1 + \alpha_2, r_2 = \alpha_2, r_3 = -\alpha_1, r_4 = -\alpha_1 - \alpha_2, r_5 = -\alpha_2$ and consider the indices of r as elements in $\mathbb{Z}/6$. Notice that $r_i + r_{i+2} = r_{i+1}, r_i + r_{i+3} = 0$ and $r_i + r_{i+4} = r_{i+5}$. The 3-chains are spanned by chains $[\alpha | \beta | \gamma]$ with the property that $\alpha + \beta \in \Phi_0, \beta + \gamma \in \Phi_0$ and $\alpha + \beta + \gamma \in \Phi_0$. The complete set $T_3(A_2)$ is given by

$$\begin{aligned}
a_i &= [r_i | r_{i+2} | r_{i+4}], \\
b_i &= [r_i | r_{i+2} | r_{i+5}], \\
c_i &= [r_i | r_{i+3} | r_{i+5}], \\
d_i &= [r_i | r_{i+3} | r_{i+1}] = -\hat{\rho}^3 b_i, \\
e_i &= [r_i | r_{i+4} | r_{i+1}] = -\hat{\rho}^3 c_i, \\
f_i &= [r_i | r_{i+4} | r_{i+2}] = -\hat{\rho}^3 a_i, \quad i \in \mathbb{Z}/6.
\end{aligned}$$

Let us define an antisymmetric 2-cochain $\omega_-^2 \in C_-^2(A_2, M)$ by

$$\omega_-^2(r_i | r_{i+2}) = 1, \quad \omega_-^2(r_i | r_{i+3}) = 0, \quad i \in \mathbb{Z}/6.$$

Figure 2: An antisymmetric 2-cocycle on A_2 . Chains sent to values 1, -1 and 0 are depicted in red, blue and gray respectively.



Then $\omega_-^2 \in Z_-^2(A_2, M)$. Indeed,

$$\begin{aligned}
 d^2 \omega_-^2(a_i) &= d^2 \omega_-^2(r_i | r_{i+2} | r_{i+4}) \\
 &= \omega_-^2(r_{i+2} | r_{i+4}) - \omega_-^2(r_{i+1} | r_{i+4}) + \omega_-^2(r_i | r_{i+3}) - \omega_-^2(r_i | r_{i+2}) \\
 &= 1 - 0 + 0 - 1 = 0, \\
 d^2 \omega_-^2(b_i) &= d^2 \omega_-^2(r_i | r_{i+2} | r_{i+5}) \\
 &= \omega_-^2(r_{i+2} | r_{i+5}) - \omega_-^2(r_{i+1} | r_{i+5}) + \omega_-^2(r_i | 0) - \omega_-^2(r_i | r_{i+2}) \\
 &= 0 - (-1) + 0 - 1 = 0, \\
 d^2 \omega_-^2(c_i) &= d^2 \omega_-^2(r_i | r_{i+3} | r_{i+5}) \\
 &= \omega_-^2(r_{i+3} | r_{i+5}) - \omega_-^2(0 | r_{i+5}) + \omega_-^2(r_i | r_{i+4}) - \omega_-^2(r_i | r_{i+3}) \\
 &= 1 - 0 - 1 - 0 = 0.
 \end{aligned}$$

That $d^2 \omega_-^2$ vanishes at the remaining chains now follows by Lemma 4.1, e.g.

$$d^2 \omega_-^2(d_i) = d^2 \omega_-^2(-\hat{\rho}^3 b_i) = -d^2 \hat{\rho}^3 \omega_-^2(b_i) = -\hat{\rho}^2 d^2 \omega_-^2(b_i) = 0.$$

This example shows the non obvious fact that $Z_-^2(A_2, M) = H_-^2(A_2, M) \neq 0$.

For our purpose it is sufficient to work with $C_+^\bullet(\Phi, M)$. Notice that the usual splitting into symmetric and antisymmetric is not immediately applicable here, since it involves division by 2, which means taking square roots in the multiplicative context (see Example 6.1).

4.1 Reversal symmetry and the cup product

Lemma 4.2. $\hat{\rho}^{p+q} \omega^p \cup \omega^q = (-1)^{\kappa_{p+q} + \kappa_p + \kappa_q} \hat{\rho}^q \omega^q \cup \hat{\rho}^p \omega^p$.

Proof.

$$\begin{aligned}
 \hat{\rho}^{p+q} \omega^p \cup \omega^q(r_1 | \dots | r_{p+q}) &= (-1)^{\kappa_{p+q}} \omega^{p+q}(r_{p+q} | \dots | r_1) \\
 &= (-1)^{\kappa_{p+q}} \omega^p(r_{p+q} | \dots | r_{q+1}) \omega^q(r_q | \dots | r_1) \\
 &= (-1)^{\kappa_{p+q} + \kappa_p + \kappa_q} \hat{\rho}^p \omega^p(r_{q+1} | \dots | r_{p+q}) \hat{\rho}^q \omega^q(r_1 | \dots | r_q) \\
 &= (-1)^{\kappa_{p+q} + \kappa_p + \kappa_q} \hat{\rho}^q \omega^q \cup \hat{\rho}^p \omega^p(r_1 | \dots | r_{p+q})
 \end{aligned}$$

□

Lemma 4.3. $\hat{\rho}^{p+q}\omega^p \cup \omega^q = (-1)^{pq+1}\hat{\rho}^q\omega^q \cup \hat{\rho}^p\omega^p.$

Proof. We compute

$$\begin{aligned}
\hat{\rho}^{p+q}\omega^p \cup \omega^q(r_1 | \dots | r_{p+q}) &= (-1)^{\kappa_{p+q}}\omega^{p+q}(r_{p+q} | \dots | r_1) \\
&= (-1)^{\kappa_{p+q}}\omega^p(r_{p+q} | \dots | r_{q+1})\omega^q(r_q | \dots | r_1) \\
&= (-1)^{\kappa_{p+q}+\kappa_p+\kappa_q}\hat{\rho}^p\omega^p(r_{q+1} | \dots | r_{p+q})\hat{\rho}^q\omega^q(r_1 | \dots | r_q) \\
&= (-1)^{\kappa_{p+q}+\kappa_p+\kappa_q}\hat{\rho}^q\omega^q \cup \hat{\rho}^p\omega^p(r_1 | \dots | r_{p+q}).
\end{aligned}$$

The proof follows by noticing that $\kappa_{p+q} + \kappa_p + \kappa_q \equiv pq + 1 \pmod{2}$. □

5 Symmetries of the root system

In this section we show that the (co)homology is equivariant with respect to automorphisms of the root system.

Let $\sigma \in \text{Aut}(\Phi)$. Then $\sigma(\beta + \gamma) = \sigma(\beta) + \sigma(\gamma)$. We define $\sigma[r_1 | \dots | r_n] = [\sigma r_1 | \dots | \sigma r_n]$, and by linear extension this defines an action on the chains $C_\bullet(\Phi)$.

Lemma 5.1. *The action of $\text{Aut}(\Phi)$ on $C_\bullet(\Phi)$ commutes with the differential ∂ .*

Proof. By a straightforward computation $\sigma\partial^n[r_0 | \dots | r_n] = \partial^n\sigma[r_0 | \dots | r_n]$. □

Lemma 5.2. *The action of $\text{Aut}(\Phi)$ on the cochains $C^\bullet(\Phi, M)$ commutes with the differential d .*

Proof. Again a straightforward computation

$$\begin{aligned}
\sigma d^n \omega^n(r_0 | \dots | r_n) &= d^n \omega^n(\sigma^{-1}(r_0 | \dots | r_n)) \\
&= \omega^n(\partial^n \sigma^{-1}(r_0 | \dots | r_n)) \\
&= \omega^n(\sigma^{-1} \partial^n(r_0 | \dots | r_n)) \\
&= \sigma \omega^n(\partial^n(r_0 | \dots | r_n)) \\
&= d^n \sigma \omega^n(r_0 | \dots | r_n)
\end{aligned}$$

proves the statement. □

6 The second cohomology group $H_+^2(\Phi, M)$

6.1 Acyclicity

In this section we show that all symmetric 2-cocycles are integrable.

Lemma 6.1. *Consider a reduced root system Φ with positive roots Φ^+ and base Δ . Addition in the ambient Euclidean space is denoted by $+$ and (\cdot, \cdot) is the invariant inner product.*

1. *If the angle between two nonproportional roots is strictly obtuse, then their sum is a root.*

2. If $\alpha, \alpha' \in \Delta$, $\alpha \neq \alpha'$, then $(\alpha, \alpha') \leq 0$ and $\alpha - \alpha' \notin \Phi$.
3. If $\beta \in \Phi^+ \setminus \Delta$, then there exists $\alpha \in \Delta$ such that $(\beta, \alpha) > 0$. In particular $\beta - \alpha \in \Phi$.
4. Let α_i, α_j be distinct simple roots. If $\beta, \gamma_i = \beta - \alpha_i$ and $\gamma_j = \beta - \alpha_j$ are roots or zero, then so is $\delta = \beta - \alpha_i - \alpha_j$.
5. If two roots β and β' add up to a positive root that is not simple, then there exists a simple root α such that $\beta + \beta' - \alpha$ is a root and either $\beta - \alpha$ or $\beta' - \alpha$ is a root as well.

Proof. Statements 1, 2 and 3 are well known and can be found in standard texts covering root systems, e.g. [7, 1]. Statement 4 can be proved as follows. Since $\beta \neq 0$

$$0 < (\beta, \beta) = (\alpha_i + \gamma_i, \alpha_j + \gamma_j) = (\alpha_i, \alpha_j) + (\alpha_i, \gamma_j) + (\gamma_i, \alpha_j) + (\gamma_i, \gamma_j)$$

and because $(\alpha_i, \alpha_j) \leq 0$ it follows that

$$(\alpha_i, \gamma_j) + (\gamma_i, \alpha_j) + (\gamma_i, \gamma_j) > 0.$$

But we also know that $\gamma_i - \gamma_j = \alpha_j - \alpha_i$ is not a root, hence $(\gamma_i, \gamma_j) \leq 0$ and

$$(\alpha_i, \gamma_j) + (\gamma_i, \alpha_j) > 0.$$

Therefore at least one of these terms is positive and at least one of $\gamma_i - \alpha_j = \delta$ and $\gamma_j - \alpha_i = \delta$ is a root or zero (by statement 1) as desired. Finally, to prove statement 5 one observes that by statement 3 there is a simple root α such that $0 < (\beta + \beta', \alpha) = (\beta, \alpha) + (\beta', \alpha)$. Hence at least one of the inner products on the right hand side is positive and therefore the involved roots can be subtracted in Φ , by statement 1. \square

Lemma 6.2. If $\omega^2 \in Z^2(\Phi, M)$ then

$$\omega^2(\alpha | -\alpha) = \omega^2(-\alpha | \alpha) \tag{2}$$

$$\omega^2(\alpha | \beta) = \omega^2(\alpha | -\alpha) - \omega^2(-\alpha | \alpha + \beta) \tag{3}$$

$$= \omega^2(\beta | -\beta) - \omega^2(\alpha + \beta | -\beta) \tag{4}$$

for all $[\alpha | \beta] \in T_2(\Phi)$.

Proof. If $[\alpha | \beta] \in T_2(\Phi)$ then

$$[\alpha | -\alpha | \alpha], \quad [\alpha | -\alpha | \alpha + \beta], \quad [\alpha | \beta | -\beta],$$

are in $T_3(\Phi)$. Equations (2)-(4) are rearrangements of $d^2\omega^2$ evaluated in these 3-chains respectively.

$$\begin{aligned} 0 &= d^2\omega^2(\alpha | -\alpha | \alpha) \\ &= \omega^2(-\alpha | \alpha) - \omega^2(0 | \alpha) + \omega^2(\alpha | 0) - \omega^2(\alpha | -\alpha) \\ &= \omega^2(-\alpha | \alpha) - \omega^2(\alpha | -\alpha), \\ 0 &= d^2\omega^2(\alpha | -\alpha | \alpha + \beta) \\ &= \omega^2(-\alpha | \alpha + \beta) - \omega^2(0 | \alpha + \beta) + \omega^2(\alpha | \beta) - \omega^2(\alpha | -\alpha) \\ &= \omega^2(-\alpha | \alpha + \beta) + \omega^2(\alpha | \beta) - \omega^2(\alpha | -\alpha) \\ 0 &= d^2\omega^2(\alpha | \beta | -\beta) \\ &= \omega^2(\beta | -\beta) - \omega^2(\alpha + \beta | -\beta) + \omega^2(\alpha | 0) - \omega^2(\alpha | \beta) \\ &= \omega^2(\beta | -\beta) - \omega^2(\alpha + \beta | -\beta) - \omega^2(\alpha | \beta) \end{aligned}$$

\square

Notice that identity (2) implies that an antisymmetric 2-cocycle vanishes on pairs of opposite roots. In terms of the Lie algebra model of Corollary 1.1 equation (2) is the symmetry of the Killing form.

Lemma 6.3. *If $\omega^2 \in Z^2(\Phi, M)$ is such that $\omega^2(\alpha|\beta) = 0$ if $\alpha, \beta \in \Phi^+$ or $\beta = -\alpha$, then $\omega^2 = 0$.*

Proof. We first show that $\omega^2(\alpha|\beta) = 0$ if $\alpha + \beta \in \Phi_0^+$. This holds by assumption when $\alpha, \beta \in \Phi^+$ or $\alpha + \beta = 0$. If on the other hand $\alpha \in \Phi^-$ that is, if α is a negative root, then $\beta \in \Phi^+$ and equation (3) shows $\omega^2(\alpha|\beta) = \omega^2(\alpha|-\alpha) - \omega^2(-\alpha|\alpha + \beta) = 0$ since the components of the argument $[\alpha|-\alpha]$ are opposite and the components of the argument $[-\alpha|\alpha + \beta]$ are both positive. Now suppose $\alpha \in \Phi^+$ and $\beta \in \Phi^-$. Then equation (4) shows $\omega^2(\alpha|\beta) = 0$, and we conclude $\omega^2(\alpha|\beta) = 0$ if $\alpha + \beta \in \Phi_0^+$.

Using this information we see from equation (3) that $\omega^2(\alpha|\beta) = 0$ if $\beta \in \Phi^+$ and from (4) that $\omega^2(\alpha|\beta) = 0$ if $\alpha \in \Phi^+$. That leaves only one case to check: $\alpha, \beta \in \Phi^-$, which again follows readily from either of these equations. \square

Theorem 6.1. *Let $\omega_+^2 \in Z_+^2(\Phi, M)$. Define $\omega^1 \in C^1(\Phi, \mathbb{Z})$ inductively on the height of the roots as follows.*

1. *The values on simple roots, $\omega^1(\alpha)$, $\alpha \in \Delta$, are chosen as free variables, in M .*
2. *If $\beta \in \Phi^+$ there exists a simple root α such that $\beta - \alpha \in \Phi_0$ and we define*

$$\omega^1(\beta) = \omega^1(\beta - \alpha) - \omega_+^2(\alpha|\beta - \alpha) + \omega^1(\alpha).$$

3. *If $\beta \in \Phi^-$ set $\omega^1(\beta) = \omega_+^2(\beta|-\beta) - \omega^1(-\beta)$.*

Then ω^1 is well defined and $d^1\omega^1 = \omega_+^2$. In particular, $H_+^2(\Phi, M)$ is trivial.

Proof. It is possible that there are multiple ways to write a positive root β of height $h + 1$ as a sum of roots of height h and 1. Therefore one needs to show that ω^1 given in the theorem is well defined. Let $\beta = \gamma_i + \alpha_i = \gamma_j + \alpha_j$ be two such decompositions. Then $\delta = \gamma_i - \alpha_j = \gamma_j - \alpha_i = \beta - \alpha_i - \alpha_j$ is a root as well, by Lemma 6.1, statement 4.

Suppose ω^1 is well defined on roots up to height h . Let $\omega_i^1(\beta) = \omega^1(\beta - \alpha_i) - \omega_+^2(\alpha_i|\beta - \alpha_i) + \omega^1(\alpha_i)$. Then

$$\begin{aligned} \omega_i^1(\beta) - \omega_j^1(\beta) &= \omega^1(\gamma_i) - \omega_+^2(\alpha_i|\gamma_i) + \omega^1(\alpha_i) - [\omega^1(\gamma_j) - \omega_+^2(\alpha_j|\gamma_j) + \omega^1(\alpha_j)] \\ &= \omega^1(\delta + \alpha_j) - \omega_+^2(\alpha_i|\gamma_i) + \omega^1(\alpha_i) - [\omega^1(\delta + \alpha_i) - \omega_+^2(\alpha_j|\gamma_j) + \omega^1(\alpha_j)] \\ &= \omega^1(\delta) - \omega_+^2(\alpha_j|\delta) + \omega^1(\alpha_j) - \omega_+^2(\alpha_i|\gamma_i) + \omega^1(\alpha_i) \\ &\quad - [\omega^1(\delta) - \omega_+^2(\alpha_i|\delta) + \omega^1(\alpha_i) - \omega_+^2(\alpha_j|\gamma_j) + \omega^1(\alpha_j)] \\ &= -\omega_+^2(\alpha_j|\delta) - \omega_+^2(\alpha_i|\gamma_i) + \omega_+^2(\alpha_i|\delta) + \omega_+^2(\alpha_j|\gamma_j) \\ &\stackrel{s}{=} \omega_+^2(\delta|\alpha_i) - \omega_+^2(\gamma_i|\alpha_i) + \omega_+^2(\alpha_j|\gamma_j) - \omega_+^2(\alpha_j|\delta) \\ &= d^2\omega_+^2(\alpha_j|\delta|\alpha_i) = 0. \end{aligned}$$

The symbol $\stackrel{s}{=}$ marks the place where the required symmetry of ω is used.

Now we turn to the proof of $d^1\omega^1(\alpha|\beta) = \omega_+^2(\alpha|\beta)$, where we will use induction on height α + height β . If $H = \max\{\text{height } \alpha + \text{height } \beta \mid \alpha, \beta, \alpha + \beta \in \Phi^+\}$ then for each $2 \leq h \leq H$ there exists $[\alpha|\beta] \in T_2(\Phi)$ with $\alpha, \beta \in \Phi^+$ and height α + height $\beta = h$. This follows from Lemma 6.1, statement 5.

First consider two simple roots, α_i and α_j . By definition $\omega^1(\alpha_j + \alpha_i) = \omega^1(\alpha_j) - \omega_+^2(\alpha_i | \alpha_j) + \omega^1(\alpha_i)$, hence

$$\begin{aligned} d^1 \omega^1(\alpha_i | \alpha_j) &= \omega^1(\alpha_j) - \omega^1(\alpha_i + \alpha_j) + \omega^1(\alpha_i) \\ &= \omega^1(\alpha_j) - [\omega^1(\alpha_j) - \omega_+^2(\alpha_i | \alpha_j) + \omega^1(\alpha_i)] + \omega^1(\alpha_i) \\ &= \omega_+^2(\alpha_i | \alpha_j). \end{aligned}$$

Suppose that $d^1 \omega^1(\tilde{\alpha} | \tilde{\beta}) = \omega_+^2(\tilde{\alpha} | \tilde{\beta})$ if $\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha} + \tilde{\beta} \in \Phi^+$ and $\text{height } \tilde{\alpha} + \text{height } \tilde{\beta} \leq h$. Now consider another pair $\alpha, \beta, \alpha + \beta \in \Phi^+$ such that $\text{height } \alpha + \text{height } \beta = h + 1$. By Lemma 6.1 statement 5 there is a simple root α_i such that $\alpha + \beta - \alpha_i \in \Phi^+$ and without loss of generality we may assume $\bar{\alpha} = \alpha - \alpha_i \in \Phi_0^+$ (notice that a positive root minus a simple root is a nonnegative root, since each root is decomposed into simple roots with all positive or all negative coefficients). In the following calculation we use that, by definition, $\omega^1(\alpha_i + \bar{\alpha} + \beta) = \omega^1(\bar{\alpha} + \beta) - \omega_+^2(\alpha_i | \bar{\alpha} + \beta) + \omega^1(\alpha_i)$ and $\omega^1(\alpha_i + \bar{\alpha}) = \omega^1(\bar{\alpha}) - \omega_+^2(\alpha_i | \bar{\alpha}) + \omega^1(\alpha_i)$, and by induction hypothesis, $\omega_+^2(\bar{\alpha} | \beta) = \omega^1(\beta) - \omega^1(\bar{\alpha} + \beta) + \omega^1(\bar{\alpha})$.

We compute

$$\begin{aligned} d^1 \omega^1(\alpha | \beta) - \omega_+^2(\alpha | \beta) &= \omega^1(\beta) - \omega^1(\alpha + \beta) + \omega^1(\alpha) - \omega_+^2(\alpha | \beta) \\ &= \omega^1(\beta) - \omega^1(\alpha_i + \bar{\alpha} + \beta) + \omega^1(\alpha_i + \bar{\alpha}) - \omega_+^2(\alpha | \beta) \\ &= \omega^1(\beta) - [\omega^1(\bar{\alpha} + \beta) - \omega_+^2(\alpha_i | \bar{\alpha} + \beta) + \omega^1(\alpha_i)] \\ &\quad + [\omega^1(\bar{\alpha}) - \omega_+^2(\alpha_i | \bar{\alpha}) + \omega^1(\alpha_i)] - \omega_+^2(\alpha | \beta) \\ &= [\omega^1(\beta) - \omega^1(\bar{\alpha} + \beta) + \omega^1(\bar{\alpha})] + \omega_+^2(\alpha_i | \bar{\alpha} + \beta) - \omega_+^2(\alpha_i | \bar{\alpha}) - \omega_+^2(\alpha | \beta) \\ &= \omega_+^2(\bar{\alpha} | \beta) - \omega_+^2(\alpha | \beta) + \omega_+^2(\alpha_i | \bar{\alpha} + \beta) - \omega_+^2(\alpha_i | \bar{\alpha}) \\ &= d^2 \omega^2(\alpha_i | \bar{\alpha} | \beta) = 0, \end{aligned}$$

thus $d^1 \omega^1 - \omega_+^2$ is zero on $(\Phi^+ | \Phi^+) \cap T_2(\Phi)$ and it also maps $[\alpha | -\alpha]$ to zero for all $\alpha \in \Phi$. Lemma 6.3 then implies $d^1 \omega^1 = \omega_+^2$. \square

6.2 Some illustrative examples

From a computational point of view [14, 10], one would very much like to work over $M[\mathbb{I}]$ instead of $M(\mathbb{I})$. The analogue of Theorem 6.1, however, does not hold, as the following example shows, which also suggests possible alternative solutions to this problem by extension of the module. We change the notation back to multiplicative, to stay close to the application we have in mind. Notice that $M[\mathbb{I}]$ is not a \mathbb{Z} -module in the sense of Section 2, since there are no multiplicative inverses (where, just to be very clear on this, multiplicative here means \mathbb{Z} -additive).

Example 6.1 ($H_+^2(B_2, M[\mathbb{I}]) \neq 1$). Consider the root system B_2 . In contrast to A_2 , in which all roots have the same length, in B_2 the roots admit two lengths, and thus are either short or long (see the Figure 3). Define $\omega^1 \in C^1(B_2, M[\mathbb{I}])$ by

$$\omega^1(\alpha) = \begin{cases} \mathbb{I} & \text{if } \alpha \text{ is a short root} \\ 1 & \text{if } \alpha \text{ is a long root.} \end{cases}$$

The coboundary $d^1 \omega^1$ takes values in $\{1, \mathbb{I}^2\}$. Therefore we can define a 2-cocycle ω_+^2 by

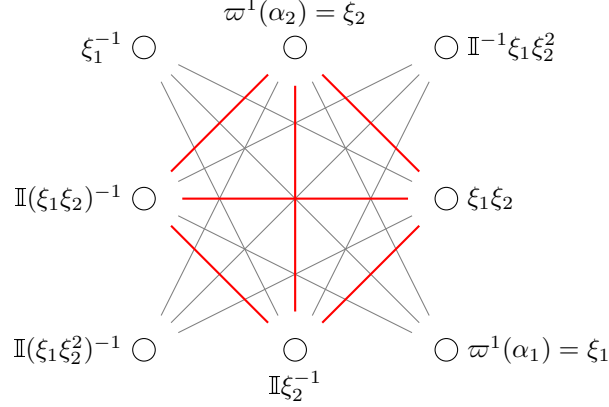
$$\omega_+^2 = \sqrt{d^1 \omega^1} \in Z^2(B_2, M[\mathbb{I}]).$$

More explicitly,

$$\omega_+^2(\alpha | \beta) = \begin{cases} \mathbb{I} & \text{if } \alpha \text{ and } \beta \text{ are short} \\ 1 & \text{otherwise.} \end{cases}$$

We integrate ω_+^2 using the algorithm of Theorem 6.1. Let α_1 be the long simple root and α_2 the short simple root of B_2 . We take the values of ϖ^1 on these simple roots as free variables (using the kernel of d^1 in full), $\varpi^1(\alpha_1) = \xi_1$ and $\varpi^1(\alpha_2) = \xi_2$. Then $\varpi^1(\alpha_1 + \alpha_2) = \varpi^1(\alpha_1)\omega_+^2(\alpha_2|\alpha_1)^{-1}\varpi^1(\alpha_2) = \xi_1\xi_2$, and $\varpi^1((\alpha_1 + \alpha_2) + \alpha_2) = \varpi^1(\alpha_1 + \alpha_2)\omega_+^2(\alpha_2|\alpha_1 + \alpha_2)^{-1}\varpi^1(\alpha_2) = \xi_1\xi_2^2\mathbb{I}^{-1}$.

Figure 3: A symmetric 2-cocycle which is not exact over $M[\mathbb{I}]$. Chains sent to \mathbb{I} and 1 are depicted in red and gray respectively.



This example is of particular interest because the 2-cocycle ω_+^2 takes values in $M[\mathbb{I}]$ but it cannot be integrated in this \mathbb{N}_0 -module. Indeed, ϖ^1 takes values ξ_1 and ξ_1^{-1} , hence requiring nonnegative powers forces $\xi_1 = 1$. The other opposite long roots then have values $(\mathbb{I}^{-1}\xi_2^2)^{\pm 1}$, hence avoiding negative powers requires $\xi_2 = \sqrt{\mathbb{I}}$. We find that there is a unique solution to $\omega_+^2 = d^1\varpi^1$ with values in $M[\sqrt{\mathbb{I}}]$, namely $\varpi^1 = \sqrt{\omega^1}$. Notice that the occurrence of $\sqrt{\mathbb{I}}$ is not in conflict with Theorem 6.1, since it would be no problem to integrate allowing negative powers, but a consequence of our abuse of the freedom given by $\ker d^1$.

Example 6.2 (Explicit integration of an element of $Z_+^2(B_2, M[\mathbb{I}, \mathbb{J}])$). The example is the ALiA obtained by starting with the 5-dimensional irreducible representation \mathbb{Y}_8 of the icosahedral group \mathbb{Y} , and considering equivariant rational maps taking values in $\mathfrak{so}(\mathbb{Y}_8)$, with poles in the smallest orbit (cf. [10]). The group generator of order five, r with $r^5 = 1$, has been diagonalised. We have computed the ALiA, computed a Cartan-Weyl basis and found that the structure constants are integer multiples of elements in $M[\mathbb{I}, \mathbb{J}]$, where \mathbb{I} and \mathbb{J} are the automorphic functions with poles in the smallest orbit and zeros in the other two exceptional orbits respectively. We then find the following element of $Z_+^2(B_2, M[\mathbb{I}, \mathbb{J}])$:

$$\begin{array}{ll} \omega_+^2(\alpha_1|\alpha_2) = 1 & \omega_+^2(-\alpha_2|-(\alpha_1 + \alpha_2)) = \mathbb{I} \\ \omega_+^2(\alpha_1 + 2\alpha_2|-\alpha_2) = \mathbb{I} & \omega_+^2(\alpha_2|-\alpha_2) = \mathbb{I} \\ \omega_+^2(\alpha_2|\alpha_1 + \alpha_2) = 1 & \omega_+^2(\alpha_2|-(\alpha_1 + \alpha_2)) = \mathbb{I} \\ \omega_+^2(\alpha_2|-(\alpha_1 + 2\alpha_2)) = 1 & \omega_+^2(\alpha_1 + \alpha_2|-\alpha_2) = \mathbb{I} \\ \omega_+^2(\alpha_1|-\alpha_1) = \mathbb{J} & \omega_+^2(\alpha_1|-(\alpha_1 + \alpha_2)) = \mathbb{J} \\ \omega_+^2(\alpha_1 + \alpha_2|-\alpha_1) = \mathbb{J} & \omega_+^2(\alpha_1 + 2\alpha_2|-(\alpha_1 + \alpha_2)) = \mathbb{I}\mathbb{J} \\ \omega_+^2(\alpha_1 + \alpha_2|-(\alpha_1 + \alpha_2)) = \mathbb{I}\mathbb{J} & \omega_+^2(\alpha_1 + \alpha_2|-(\alpha_1 + 2\alpha_2)) = \mathbb{J} \\ \omega_+^2(\alpha_1 + 2\alpha_2|-(\alpha_1 + 2\alpha_2)) = \mathbb{I}\mathbb{J} & \omega_+^2(-\alpha_1|-\alpha_2) = 1 \end{array}$$

This is integrated to

<i>long</i>	<i>short</i>
$\omega^1(\alpha_1) = 1$	$\omega^1(\alpha_2) = 1$
$\omega^1(-\alpha_1) = \mathbb{J}$	$\omega^1(-\alpha_2) = \mathbb{I}$
$\omega^1(\alpha_1 + 2\alpha_2) = 1$	$\omega^1(\alpha_1 + \alpha_2) = 1$
$\omega^1(-(\alpha_1 + 2\alpha_2)) = \mathbb{I}\mathbb{J}$	$\omega^1(-(\alpha_1 + \alpha_2)) = \mathbb{I}\mathbb{J}$

The short count (summing over the Killing forms $K_{\tilde{\mathfrak{g}}}(\tilde{e}_{+\alpha}, \tilde{e}_{-\alpha})$, $\alpha = \alpha_2, \alpha_1 + \alpha_2$) is $\mathbb{I} + \mathbb{I}\mathbb{J}$ (cf. [10]), the long count ($\alpha = \alpha_1, \alpha_1 + 2\alpha_2$) is $\mathbb{J} + \mathbb{I}\mathbb{J}$. The total count (3 \mathbb{I} s and 3 \mathbb{J} s) is in accordance with the predictions given by the codimensions of the invariants subspaces under the conjugating action of the generators of the group on \mathfrak{so}_5 .

In the theory of ALiAs several models of the ALiA play a role: the invariant matrices, the matrices of invariants and the integrated model as the one above. The matrices of invariants, once in Weyl-Chevalley normal form, are always natural monomial in \mathbb{I} and \mathbb{J} , and they can be used in the subsequent search for integrable systems, which was the original motivation for ALiAs (see e.g. [13, 11, 12, 2]). The integrated models cannot in general be used for this purpose, unless they are natural, but they play a role in establishing whether two given ALiAs are isomorphic or not, and in the choice of a normal form for isomorphic cases.

7 Conclusions

We have shown in this paper how the cohomology of root systems appears naturally in the theory of ALiAs, once a Cartan-Weyl basis of the ALiA is computed. The theory of ALiAs, and more specifically, their normal form theory, has been developed in the last decennium and this explains why there is no mention of cohomology of root systems in the literature, since it is the symmetric case that appears in a natural way and there does not exist an analogous theory in the skew symmetric case, although there are some developments in the theory of Kac-Moody algebras that remind one of such a theory (cf. [8, p. 105]).

Even though the cohomology theory of root systems is new, it is naturally connected to classical theory of root systems and representations: using the invariant inner product on the dual of the Cartan subalgebra, linear functions on this vector space are identified with elements of it. Thus, integer valued 1-cocycles $Z^1(\Phi, \mathbb{Z})$ are identified with the coweight lattice of the associated Lie algebra. Moreover, taking the quotient with the action of the Weyl group \mathcal{W} results in a one-to-one correspondence

$$Z^1(\Phi, \mathbb{Z})/\mathcal{W} \leftrightarrow \{\text{irreducible representations of } \mathfrak{g}(\Phi^\vee)\}$$

where Φ^\vee is the dual root system. The details of this construction, including the modular case where the symmetry group is affine, will be published elsewhere.

We formulate a number of questions for future research.

Question 1. Under what conditions can $\omega^2 \in Z_+^2(\Phi, \mathbb{M}[\mathbb{I}])$ be integrated to $\omega^1 \in C^1(\Phi, \mathbb{M}[\mathbb{I}])$? In other words, what is $H_+^2(\Phi, \mathbb{M}[\mathbb{I}])$?

Question 2. Under what conditions is $\omega^1 \in C^1(\Phi, \mathbb{M}[\mathbb{I}])$ differentiated into $C_+^2(\Phi, \mathbb{M}[\mathbb{I}])$?

Question 3. Under what conditions is $\omega^1 \in C^1(\Phi, \mathbb{M}(\mathbb{I}))$ differentiated into $C_+^2(\Phi, \mathbb{M}[\mathbb{I}])$?

Question 4. Under what conditions can $\omega^2 \in Z_+^2(\Phi, \mathbb{M}[\mathbb{I}])$ be integrated to $\omega^1 \in C^1(\Phi, \mathbb{M}[\mathbb{I}^{1/n}])$?

At the moment the most direct application of the cohomology theory is the possibility to experiment with rational monomial models in the hope of obtaining (by differentiation) natural monomial Lie algebras with given Cartan matrix. We remark that such algebras can be seen as deformations of Lie algebras over \mathbb{C} . The existence of such deformations is, in the semisimple case, not in contradiction with the fact that the second Lie algebra cohomology is trivial, since this argument implicitly assumes that one is working over a field, not a semiring (like the natural monomials).

Considering the importance of infinite dimensional Lie algebras in both physics and mathematics, we hope that this approach will lead to new and interesting developments.

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